Defining the Characteristics of Critical Mathematical Thinking

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In this paper, we report on the interim findings of a study that seeks to identify the characteristics of children's Critical Mathematical Thinking CMT. Characteristics of CMT were initially generated from a synthesis of relevant research literature and then validated using a case study methodology via trials in early childhood classrooms. This paper provides a framework for CMT distilled from the literature and an illustrative case study of one student to provide tentative evidence that young children's use of CMT capabilities can be identified. The long term aim of this line of research is to explore the potential to promote CMT capabilities in a targeted manner.

It has been well established, that children begin to use mathematical thinking skills from a young age (e.g., Bobis, Clarke, Clarke, Thomas, Young-Loveridge, & Gould, 2005; Doig & Ompok, 2010). Evidence for this claim is primarily found in studies that have looked at the way young children learn mathematics (Sarama & Clements, 2009; Clarke, Clarke & Roche, 2011). Consistent with this perspective, current advice about the development of mathematical thinking capabilities in early learning contexts is that instruction should adopt an investigative approach to promote deep understanding and connections between mathematical ideas (Clements, 2001; Sarama, Lange, Clements, & Wolfe, 2012). The need to adopt investigative approaches, as a means of promoting mathematical thinking capabilities, is further supported by curriculum documents and educational policy (ACARA, 2016; Australian Government Department of Education Employment and Workplace, 2009). As a result, educators of young learners have worked to create mathematical learning experiences that focus on open-ended approaches to support creativity, imagination and reflexivity in addition to conceptual development. While early childhood educators have been provided with direction on the content and processes to be taught, as well appropriate pedagogical approaches, there has been limited advice from research literature about how to make judgments about levels of development for students' mathematical thinking capabilities. As such, limited means is available to assist early childhood teachers in identifying and describing students' mathematical thinking capabilities. It is essential that tools for assessing the capabilities be developed, in order to provide feedback to students and teachers about a student's progress – informing teachers' decisions about appropriate approaches to instruction.

The purpose of this paper is to outline a framework within which the characteristics of children's critical mathematical thinking are outlined and described. Consistent with this purpose, we will address the following research question.

What are the observable characteristics of young children's critical mathematical thinking?

In attending to this question, we will (1) provide a synthesis of current literature related to children's critical mathematical thinking; (2) extend the synthesis to define *critical*

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mathematical thinking; (3) present a case study as an illustrative example of categories within a framework for critical mathematical thinking; and (4) discuss the potential for further research.

Mathematical Thinking

Advancing children's mathematical thinking has been a focus of an expanding body of research in recent years (e.g., Carpenter, Franke, Johnson, Turrou, & Wager 2017; Breen & O'Shea, 2010; Fraivillig, Murphy, & Fuson, 1999). While perspectives in the filed are wide ranging, conceptions of mathematical thinking, appear to coalesce around a number of central principles: children require mathematical knowledge (Burton 1984); a basic understanding of mathematical concepts (Burton, 1984); and opportunities to engage in mathematical learning in different ways, all within a learning environment that fosters mathematical development (Ginsburg, Cannon, Eisenband, & Pappas, 2006). M*athematical thinking* refers more to the "doing" of mathematics rather than the memorising of formulas or the application of procedures (Stein, Grover, & Henningsen, 1996) and so involves problem solving, reasoning and critical thinking. Key characteristics that demonstrate mathematical thinking have been synthesized from relevant research literature as, connecting procedures, tacking complex problems in novel ways, reasoning and sense-making (Table 1, # 1-4).

Research conducted by Cengiz, Kline, & Grant, (2011) investigated the types of tasks used to extend thinking with children in grades 1 – 4. Strategies observed by these researchers found that teachers invited children to provide an evaluation of their learning that would allow for reflection and sharing of ideas or strategies (Table 1, #5). Strategies to encourage reasoning were also researched and found that teacher probing questions such as "What makes you say that? How do you know? Why do you suppose that?" (Cengiz, Kline, & Grant, 2011) elicited children's thinking. Thus, how tasks are designed is critical for uncovering children's mathematical thinking - requiring children to reason and think mathematically (Stein Grover & Henningsen,1996). These include open-ended tasks that have multiple answers, many modes of representation and in particular the opportunity for children to explain and justify their thinking.

How tasks are implemented, however, is a critical aspect of eliciting children's mathematical thinking. A study by Fraivillig, Murphy and Fuson (1999) of first grade children and teachers looked at teacher practices and found teachers were using eliciting, supporting and extending strategies with the children to facilitate their mathematical thinking. The study found that with teachers promoting children's thinking, the following mathematical thinking capabilities emerged from the learners: describing solutions (#7); elaborating on an idea (#7); clarifying own solutions (#7); generalizing across concepts (#1); noting relationships (#1); and considerations of alternate solutions (#6). Similarities between the research examined thus far, has found that in order for children to think mathematically, teachers have a significant role as a guide, for the thinking to emerge. The characteristics of mathematical thinking is summarised in Table 1.

Table 1
Characteristics that Demonstrate Mathematical Thinking

1	Connecting procedures/ noting relationships/ generalizing across concepts	(Fraivillig, Murphey, & Fuson, 1999; Stein, Grover, & Henningsen, 1996)
2	Tackling complex problems in novel ways	(Stein, Grover, & Henningsen, 1996)

3	Reasoning	(Cengiz, Kline, & Grant, 2011; Jacobs, Lamb, & Philipp, 2010; Stein, Grover, & Henningsen, 1996)
4	Sense-making	(Jacobs, Lamb, & Philipp, 2010; Stein, Grover, & Henningsen, 1996)
5	Evaluating	(Cengiz, Kline, & Grant, 2011)
6	Considering other methods/strategies/ alternate solutions	(Cengiz, Kline, & Grant, , 2011; Fraivillig, Murphey, & Fuson, 1999)
7	Describing solutions/ Clarification of solutions/Elaborating on ideas	(Fraivillig, Murphey, & Fuson, 1999)

Critical Mathematical Thinking

Stenberg (1986) identified the construct of critical thinking as a lens to gain more in-depth insight into children's mathematical thinking. According to Sternberg, critical thinking includes "mental processes, strategies, and representations people use to solve problems, make decisions, and learn new concepts" (p.3). Additionally, critical thinking includes building knowledge, comparing and identifying differences, supporting ideas with reasons and examples and considering alternative solutions (Florea & Hurjui, 2015). The importance of critical thinking can be located in many educational documents and international assessments such as New Media Consortium (NMC)/ Consortium for School Networking (CoSN) Horizon Report: 2016 K - 12 Edition (Johnson, Adams Becker, Cummins, Estrada, Freeman, & Hall, 2016) and the Programme for International Students Assessment (PISA) (Organisation for Economic Co-operation and Development, 2018).

While mathematical thinking and critical thinking have considerable common ground, they differ at the level of detail. An alignment between mathematical thinking and critical thinking is presented in Table 2. An additional column, titled *Critical Mathematical Thinking Capabilities* provides detail of additional observable features of this alignment.

Table 2
Aligning Critical and Mathematical Thinking

Mathematical Thinking characteristics from Table 1	Capabilities of Critical Thinking		Critical Mathematical Thinking Capabilities
Connecting procedures/ noting relationships	Generate and evaluate knowledge; Apply new ideas to specific contexts	➔	Uses mathematical and other understandings to generate, evaluate, connect and create new ideas
Tackling complex problems in novel ways	Seek possibilities; Consider alternatives; Imagination; Innovation; Test	➔	Identifies and performs many ways to solve mathematical problems
Reasoning	Reason; Logic	➔	Provides reasons or judgments
Sense-making	Clarify concepts and ideas; Interpret; Sequence; Generalise	➔	Uses mathematical strategies to prove the answer is possible

Evaluating	Use evidence to support an argument; Draw reasoned conclusions; Evaluate	➔	Self-evaluates, using mathematical evidence and reasoning
Considering other methods/strategies/ alternate solutions	Solve problems; Resourcefulness; Analyse; Compare	➔	Identifies and performs many ways to solve mathematical problems
Describing solutions/ Clarification of solutions/Elaborating on ideas	Thinking skills and strategies; Explain; Question; Infer; Hypothesise; Appraise	➔	Builds on ideas through explanation, questioning, inferencing, hypothesising and appraising

The alignment between mathematical thinking and capabilities of critical thinking is displayed in Table 2 and results in an overall definition of *Critical Mathematical Thinking* (CMT). The following points provide the summary of CMT:

- Using the knowledge of mathematics and mathematical processes to:
- Use mathematical and other understandings to generate, evaluate, connect and create new ideas;
- Identify and performs many ways to solve mathematical problems;
- Provide reasons or judgments;
- Use mathematical strategies to prove the answer is possible;
- Self-evaluate, using mathematical evidence and reasoning; and
- Build on ideas through explanation, questioning, inferencing, hypothesising and appraising.

Methodological Approach

To address the research question about characteristics of young children's critical mathematical thinking, a case study methodology was used to gain insights into one child's sophisticated mathematical strategies when engaging with high-level open-ended tasks. Case study was selected as data was drawn from a bounded system (Stake, 1995); one early childhood classroom.

Participant Selection

The data for this study includes four major sources: (1) classroom observation field notes; (2) semi-structured focus group interviews; and (3) interviews of Kindergarten (1st year of formal schooling in NSW) classroom teachers; and (4) interviews of Kindergarten children. The timing of this research was during the beginning of a school year.

Four mathematics lessons, led by the teacher, were observed. These lessons included the entire class and were based on activities related to whole number patterns and algebraic thinking. The CMT capabilities listed in Table 2 was used by the researcher as a lens to identify children that presented CMT capabilities. Each child in the class was questioned by the research about their learning during the lesson by using probing or prompting questions (Rigelman, 2007) based on the CMT capabilities. Summaries of their responses were video recorded. Field notes included the observations made by the researcher during the class lesson informed by the CMT framework. Based on observation notes and in class questioning, 38 children that showed potential CMT took part in 3 separate focus groups. The focus groups were based on the mathematics learning the children were participating in during their classroom teacher led lessons. The mathematics included: patterns and algebra, addition and subtraction and two-dimensional shapes. The researcher posed questions in relation to the teacher designed mathematical task, in order to ascertain students levels of mathematical reasoning. Examples of such questions included: How did you work that out?; What would happen if…?; Is there another way to do this?.

One child, Jordan, was selected for the illustrative case study, reported here, on the basis of the high level of interest he displayed in investigative tasks during the observed lesson and the insightfulness demonstrated during follow-up focus group interviews. Jordan's interview was based on a semi-structured interview protocol that included 8 open-ended questions aimed at prompting responses indicative of critical mathematical thinking. Each question was designed to allow for specific CMT characteristics, as identified in Table 2, to emerge.

The researcher video recorded and transcribed the interview with Jordan. Each response was mapped against the CMT capabilities to determine the scope of Jordan's development in this area.

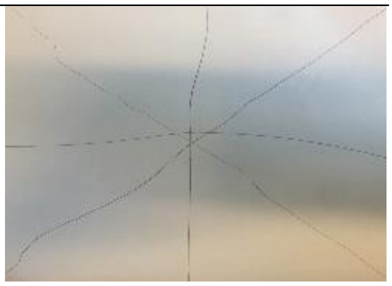
Results - Jordan's Critical Mathematical Thinking

In this section, an illustrative example is presented based on three out of eight questions from Jordan's interview. These three questions were selected for discussion because they provided clearly observable characteristics of critical mathematical thinking. Jordan's responses are mapped against the Key CMT characteristics (Table 2).

Question 1

Question 1 required Jordan to find the middle of wall to hang a picture frame. The manipulatives given to Jordan included an A3 sheet of paper and a small laminated picture frame. No additional resources were provided.

Table 3
Jordan's Responses to the Question 1 of the Semi-Structured One-On-One Assessment Instrument

CMT Question Instructions	Jordan's response	Jordan's work Sample for Question 1
This is a framed photograph of my son Joey. (Hold up real framed photograph.) I have a blank wall at home and I would like to hang this photograph in the middle of that wall. Let's imagine this A3 piece of paper (hold up A3 paper) is the blank wall and here is a smaller picture frame (hold up small picture frame). How can you find the	"Are you going to give me a ruler? You can't fold a wall so you can't fold this paper. I will draw a line (diagonally) here and another line (diagonally) here and just to prove it to you I will draw another line this way (horizontally) and another line this way (vertically), that is the middle (pointed to where the lines intersect)".	

exact place to hang up Joey's photograph?
Key CMT characteristics:
Uses mathematical and other understandings to generate, evaluate, connect and create new ideas
Identifies and performs many ways to solve mathematical problems
Provides reasons or judgments
Uses mathematical strategies to prove the answer is possible
Self-evaluates, using mathematical evidence and reasoning
Identifies and performs many ways to solve mathematical problems

After some probing from the researcher, including directions to use the A3 sheet of paper to draw or write on, Jordan was able to produce a mathematical strategy for question 1 by drawing 4 lines that intersect in the centre of the page. The alignment of Jordan's response with the CMT capabilities are as follows:

- Estimating: Determining where the midpoint of the paper was to begin to draw a line. Using informal measurement to draw cross sections
- Grasping principles/noting relationships: Using known understanding and skills of drawing lines, half, in a new context of 'centre'
- Offering opinions: Questioning if a wall can be folded or if a rule was to be provided
- Reasoning: Proving that where all lines intersected was the centre

Question 2

This task required Jordan to consider the associative property of addition. The example provided included one-digit numbers. The number sentences, 3 + 3 and 4 + 2, were typed on a card for Jordan to view.

Table 4
Jordan's Responses to the Question 2 of the Semi-Structured One-On-One Assessment Instrument

CMT Question Instructions	Jordan's response
Why is 3 + 3 the same as 4 + 2. Change the numbers to 2 digit numbers if appropriate. Can you tell me why 3 + 3 is the same as 4 + 2? Why/why not? Provide two reasons why they are equal. Can you tell me another way you can work this out?	"If you put 4 + 2, then you take one away and you put it with a 3, it is even 3+ 3=6" "They both equal 6"

The response provided by Jordan for question 2 involved the compensation strategy – subtracting from one digit and adding to another digit. Jordan displayed CMT capabilities in line with those expected of a Year 3 or 4 child, according to the Australian Curriculum: mathematics content descriptors (ACARA, 2016). Alignment of CMT capabilities displays Jordan's understandings by:

- Classifying: Arranging numbers to allow for compensation strategy discussion
- Grasping principles/noting relationships: Understandings the value of place value and applying it to a new situation
- Offering opinions with reason: Articulating that if you take one digit away and place it elsewhere it still has value

Question 3

Jordan was asked to consider a way to identify the number of tiles required to cover a floor surface. The researcher provided a photograph of a cubby house for Jordan to view and tiles that measured 10 cm x 10 cm.

Table 5
Jordan's Responses to the Question 3 of the Semi-Structured One-On-One Assessment Instrument

CMT Question Instructions	Jordan's response
I have just finished building a cubby house for my children at home (show picture of the cubby house). I would like to put these tiles down on the floor of the cubby house (show rectangular tile). How can I work out how many tiles I need?	"Well you know how big your tile is so you can see how many fit across and then you can buy one row at a time to save money".

Jordan displayed the capacity to make reference to the tile being used as a repetitive unit of measurement, to determine the number of required tiles. His reference to money provides an insight into Jordan's understanding of finances and quantity. There is also an initial reference to the concept of area with his discussion on rows by saying, "how many fit across". The CMT capabilities displayed in Jordan's responses included:

- Assuming: Considering that many will fit in a row and that there will be many rows and using the tile as an informal unit of measure
- Grasping Principles: Showing understanding of area and ways to find out the area in a real-life context
- Offering opinions with reasons: The inclusion of money provides a real-life situation

Conclusion

This paper has addressed the research question: What are some characteristics of young children's critical mathematical thinking? In the case presented here, Jordan's responses demonstrated evidence of CMT characteristics such as reasoning, noting relationships and classifying. These observed characteristics of CMT are in agreement with previous research findings that young children are capable of CMT (Bobis et al., 2005; Doig & Ompok, 2010). This study takes a small step to extending previous knowledge in the area of children's critical mathematical thinking by providing a framework that shows the potential to act as the basis for developing tools for determining students' CMT capability.

This research is an initial step in advancing understanding of young children's higher order thinking in mathematics. Thus, the study, has implications for teachers and policy makers who may review the way in which mathematical learning and assessment is designed for young children. While this study provides evidence of how a child's responses can display critical mathematical thinking and a range of mathematical strategies, no attempt is made to generalise as the purpose of the paper is to seek to understand. These initial findings, however, will inform the larger study from which the data used here was sourced and provide further scope for investigating how CMT can be assessed in young children.

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